LOCAL APPROXIMATION OF A VORTEX LAYER BY A SYSTEM OF DISCRETE VORTICES

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In aerodynamics a wing and the vortex sheet behind it are often simulated by a system of discrete vortices. A rigorous justification for this simulation in the case of a steady-state flow of an incompressible liquid around a thin curvilinear profile has been given in [1].

The idea of dividing up the bearing vortex surface around which the flow occurs, leads to the method of discrete vortices, which has been successfully used to calculate the aerodynamic characteristics of aircraft [2-5].

Theoretical schemes with a uniform distribution of discrete vortices and control points in which the boundary conditions of the corresponding boundary value problem have to be satisfied has obtained the widest application. For these schemes the convergence of the approximate solution of the one-dimensional singular integral equation with a Cauchy kernel to the accurate solution has been proved in any fixed section inside the integration interval [6, 7], and also the convergence over the whole interval with respect to the norm in L_1 [8] for all permissible classes of the solutions. At the same time it has been pointed out that the uniform theoretical scheme gives an irremovable error in the approximate solution in the region of the ends of the interval [9, 4, 6].

The use of theoretical schemes with a nonuniform distribution of the discrete vortices and control points, enabling one, in principle, to obtain a uniform approximation of the intensity of the discrete vortices to their corresponding accurate values over the whole interval, opens up new possibilities. This scheme was apparently first proposed in [9]. The question of the justification for the use of nonuniform schemes has not been investigated in practice.

It should be noted that a key feature in the problem of constructing a solution of the singular integral equations by the method of discrete vortices is a problem of the approximation of Cauchy-type integrals by a corresponding quadrature formula. This paper is devoted to an investigation of this question.

1. Consider the Cauchy-type integral

$$F(x_0) = \int_0^1 \frac{\gamma(x)\,dx}{x - x_0}, \tag{1.1}$$

defined in the interval [0, 1] of the real axis. As regards the function $\gamma(x)$, we will assume that it can be represented in the form

$$\gamma(x) = \sqrt{\frac{1-x}{x}}\,\varphi(x), \tag{1.2}$$

where the function $\varphi(x)$ satisfies the Hölder condition with index α_1 for $x \in [0, \delta]$ and α_2 for $x \in (\delta, 1]$, assuming that $0 < \alpha_p \leqslant 1$, $p = 1, 2$; $0 < \delta < 1$.

We will divide the section [0, 1] by the points $c_0 = 0, c_1, \ldots, c_n = 1$ into n elements of length $h_m = c_m - c_{m-1}$, $m = 1, \ldots, n$. We will introduce the quantities

$$\Gamma_m = \int_{c_{m-1}}^{c_m} \gamma(x)\,dx, \quad m = 1, \ldots, n \tag{1.3}$$

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and two separate points $x_1, \ldots, x_n$ and $x_{01}, \ldots, x_{0n}$, which satisfy the conditions

$$x_m \in [c_{m-1}, c_m], \; m = 1, \ldots, n;$$
$$x_{0k} \in (c_{k-1}, c_{k+1}), \; k = 1, \ldots, n-1, \; x_{0n} \in (c_{n-1}, c_n]. \tag{1.4}$$

In accordance with these conditions we will represent x_m and x_{0k} in the form

$$x_m = c_{m-1} + h_m\mu_m, \; x_{0k} = c_{k-1} + h_k\nu_k, \; m, k = 1, \ldots, n, \tag{1.5}$$

where the coefficients μ_m, ν_k can vary in the limits $0 \leqslant \mu_m \leqslant 1$, $0 < \nu_k < 2$.

We will introduce the function

$$G(x_{0k}) = \sum_{m=1}^{n} \frac{\Gamma_m}{x_m - x_{0k}}, \tag{1.6}$$

defined at the points $x_0 \in [0, 1]$, $k = 1, \ldots, n$.

We will consider the following problem: to construct a sequence of sets of points $x_{01}, \ldots, x_{0n}$ and $x_1, \ldots, x_n$, which satisfy conditions (1.4), for which the function (1.6) converges uniformly to the Cauchy-type integral (1.1) at all points $x_{0k} \in [0, 1]$ when n increases without limit.

To solve this problem we can arrange the coefficients $\mu_1, \ldots, \mu_n$ and $\nu_1, \ldots, \nu_n$ assuming the function $\gamma(x)$ and the points $c_0, c_1, \ldots, c_n$ to be given.

Note that in terms of hydrodynamics the function $\gamma(x)$ is the intensity of the vortex layer, Γ_m is the intensity of the discrete vortex in the element $[c_{m-1}, c_m]$, x_m is the coordinate of this vortex, x_{0k} is the coordinate of the control point, the function $F(x_{0k})$ defines the velocity of the liquid induced by the continuous vortex layer at the point x_{0k}, and the function $G(x_{0k})$ is the velocity induced at the same points of the discrete-vortex system.

2. We will confine ourselves to the case of a uniform splitting of the section [0, 1] into elements. Then $c_m = mh$, $h = 1/n$, $m = 0, 1, \ldots, n$. We will introduce into the section [0, 1] two regions $[0, \delta_1]$, $[\delta_2, 1]$ and a section $[0, \delta]$, assuming that $0 < \delta_2 < \delta < \delta_1 < 1$. We will denote by N_1, n_1, and N_2, respectively, the number of elements in the sections $[0, \delta_1]$, $[0, \delta]$, $[0, \delta_2]$. In accordance with (1.2), we will assume that in each region the function $\gamma(x)$ can be represented in the form

$$\gamma(x) = \Phi_1(x)/\sqrt{x}, \; \Phi_1(x) = \varphi(x)\sqrt{1-x} \quad \text{for} \quad x \in [0, \delta_1],$$
$$\gamma(x) = \Phi_2(x)\sqrt{1-x}, \; \Phi_2(x) = \varphi(x)/\sqrt{x} \quad \text{for} \quad x \in [\delta_2, 1], \tag{2.1}$$

where for the functions Φ_1, Φ_2 for all $x \in [c_{m-1}, c_m]$ the following inequalities are satisfied:

$$|\Phi_1(x) - \Phi_1(x_m)| \leqslant A_1 n^{-\alpha_1}, \quad m = 1, \ldots, N_1,$$
$$|\Phi_2(x) - \Phi_2(x_m)| \leqslant A_2 n^{-\alpha_2}, \quad m = N_2 + 1, \ldots, n. \tag{2.2}$$

Here A_1 and A_2 are positive constants. We will further put

$$M_1 = \sup_{[0,\delta_1]} |\Phi_1(x)|, \quad M_2 = \sup_{[\delta_2, 1]} |\Phi_2(x)|.$$

We will represent the functions $F(x_{0k})$, $G(x_{0k})$ in the form

$$F(x_{0k}) = \sum_{m=1}^{n} F_m(x_{0k}), \quad G_m(x_{0k}) = \sum_{m=1}^{n} G_m(x_{0k}); \tag{2.3}$$

$$F_m(x_{0k}) = \int\limits_{c_{m-1}}^{c_m} \frac{\gamma(x)\,dx}{x - x_{0k}}, \quad G_m(x_{0k}) = \frac{\Gamma_m}{x_m - x_{0k}}. \tag{2.4}$$

Taking into account Eqs. (2.4), (1.3), (2.1), and (2.2)

$$F_m(x_{0k}) = \Phi_p(x_m) f_m^{(p)}(x_{0k}) \left[1 + O\left(n^{-\alpha_p}\right)\right],$$
$$G_m(x_{0k}) = \Phi_p(x_m) g_m^{(p)}(x_{0k}) \left[1 + O\left(n^{-\alpha_p}\right)\right], \tag{2.5}$$

where the index p = 1 for m = 1, ..., N_1 and p = 2 for m = N_2 + 1, ..., n, while the functions $f_m^{(p)}$, $g_m^{(p)}$ are given by the following expressions:

$$f_m^{(1)} = \sqrt{\frac{n}{\sigma_k}} \ln \frac{(\sqrt{m}-\sqrt{\sigma_k})(\sqrt{m-1}+\sqrt{\sigma_k})}{(\sqrt{m}+\sqrt{\sigma_k})(\sqrt{m-1}-\sqrt{\sigma_k})}, \quad m \neq k, \quad k+1, \tag{2.6}$$

$$f_k^{(1)} + f_{k+1}^{(1)} = \sqrt{\frac{n}{\sigma_k}} \ln \frac{(\sqrt{\sigma_k}+\sqrt{k-1})(\sqrt{k+1}-\sqrt{\sigma_k})}{(\sqrt{\sigma_k}-\sqrt{k-1})(\sqrt{k+1}+\sqrt{\sigma_k})};$$

$$f_m^{(2)} = -\frac{2}{\sqrt{n}}\Bigg[\sqrt{n-m+1}-\sqrt{n-m}-\frac{\sqrt{\tau_k}}{2} \tag{2.7}$$

$$\times \ln \frac{(\sqrt{\tau_k}-\sqrt{n-m})(\sqrt{\tau_k}+\sqrt{n-m+1})}{(\sqrt{\tau_k}+\sqrt{n-m})(\sqrt{\tau_k}-\sqrt{n-m+1})}\Bigg], \quad m \neq k, \quad k+1,$$

$$f_k^{(2)} + f_{k+1}^{(2)} = -\frac{2}{\sqrt{n}}\left[\sqrt{n-k+1}-\sqrt{n-k-1}-\frac{\sqrt{\tau_k}}{2}\ln\frac{(\sqrt{\tau_k}-\sqrt{n-k-1})(\sqrt{n-k+1}+\sqrt{\tau_k})}{(\sqrt{\tau_k}+\sqrt{n-k-1})(\sqrt{n-k+1}-\sqrt{\tau_k})}\right], \quad k \neq n,$$

$$f_n^{(2)} = -\frac{2}{\sqrt{n}}\left[1-\frac{\sqrt{\tau_k}}{2}\ln\frac{1+\sqrt{\tau_k}}{1-\sqrt{\tau_k}}\right];$$

$$g_m^{(1)} = 2\sqrt{n}\,\frac{\sqrt{m}-\sqrt{m-1}}{m-\tau_k-1+\mu_m}, \quad g_m^{(2)} = \frac{2}{3\sqrt{n}}\,\frac{(n-m+1)^{3/2}-(n-m)^{3/2}}{m-n+\tau_k-1+\mu_m}; \tag{2.8}$$

$$\sigma_k = k-1+\nu_k, \; \tau_k = n-k+1-\nu_k. \tag{2.9}$$

Note that when deriving these equations for m = k + 1 we chose $\Phi_p(x_k)$ as the value of $\Phi_p\cdot(x_{k+1})$.

We will now investigate the difference between the functions F and G at the point x_{0k}, assuming k = 1, ..., n_1. Taking Eqs. (2.3)-(2.8) into account we have

$$F(x_{0k}) - G(x_{0k}) = \sum_{m=1}^{N_1} \Phi_1(x_m)\left[f_m^{(1)}(x_{0k}) - g_m^{(1)}(x_{0k})\right] + \sum_{m=N_1+1}^{n} \Phi_2(x_m)\left[f_m^{(2)}(x_{0k}) - g_m^{(2)}(x_{0k})\right]. \tag{2.10}$$

Here we have omitted the common factors $1 + O(n^{-\alpha_p})$ (p = 1.2) on the right side of the expression, which are unimportant in investigating the convergence of the function $G(x_{0k})$ to $F(x_{0k})$. In addition, when $\nu_k = 1$ the writing of the first sum for m = k, k + 1 is conditional, since for these values of ν_k only the sum $f_k^{(1)} + f_{k+1}^{(1)}$ has any meaning.

We will estimate each term separately in Eq. (2.10). We will first consider the first sum. Taking the above notes into account this sum can be converted to the form

$$\sum_{m=1}^{N_1} \Phi_1(x_m)\left[f_m^{(1)}(x_{0k}) - g_m^{(1)}(x_{0k})\right] = \Phi_1(x_{N_1})\,S_{N_1}^{(1)}(\sigma_k) + \sum_{r=1}^{N_1-1}{}' \left[\Phi_1(x_r) - \Phi_1(x_{r+1})\right] S_r^{(1)}(\sigma_k),$$

where the prime denotes summation over all r, apart from r = k, while

$$S_r^{(1)}(\sigma_k) = \sum_{m=1}^{r}\left[f_m^{(1)}(x_{0k}) - g_m^{(1)}(x_{0k})\right].$$

It follows from Eqs. (2.6) and (2.8) that when $|m - k| \gg 1$

$$f_m^{(1)}(x_{0k}) - g_m^{(1)}(x_{0k}) = O\left(\frac{\sqrt{n/m}}{(m-k)^2}\right).$$

This enables us to conclude that

$$\sum_{r=1}^{N_1-1}{}' \left|S_r^{(1)}(\sigma_k)\right| \leqslant \frac{B_1\sqrt{n}}{k^{\beta}},$$

where B_1 is a certain constant, and β is an arbitrary number which satisfies the condition 0 < β < 0.5. Hence, taking inequalities (2.2) into account we obtain the estimate

$$\left|\sum_{r=1}^{N_1-1}{}' \left[\Phi_1(x_r) - \Phi_1(x_{r+1})\right] S_r^{(1)}(\sigma_k)\right| \leqslant \frac{2A_1B_1}{k^{\beta}n^{\alpha_1-0.5}}. \tag{2.11}$$

We will now choose the coefficients ν_1, ..., ν_{n_1} and μ_1, ..., μ_{N_1} in such a way that $S_{N_1}^{(1)} = 0$. Then the estimate of the first sum on the right side of Eq. (2.10) agrees with the estimates (2.11).

TABLE 1

k	$\nu_k(0)$
1	0,55
2	0,52
3	0,51
4	0,51
5	0,5
$n-1$	0,5
n	0,38

The requirement $S_{N_1}^{(1)} = 0$ leads to the following transcendental equation:

$$\ln \frac{\sqrt{N_1}+\sqrt{\sigma_k}}{\sqrt{N_1}-\sqrt{\sigma_k}} - 2\sqrt{\sigma_k}\sum_{m=1}^{N_1}\frac{\sqrt{m}-\sqrt{m-1}}{\sigma_k - m + 1 - \mu_m} = 0, \quad k = 1, \ldots, n_1. \tag{2.12}$$

For specified values of the coefficients μ_m the roots of this equation are the quantities σ_k, connected with the coefficients ν_k by Eq. (2.9). It is also permissible to assign the coefficients ν_k or σ_k with a certain μ_m as solutions of the system of equations (2.12). Examples of these solutions are given below.

We will estimate the second sum in Eq. (2.10). Equations (2.7) and (2.8) enable one to obtain the following asymptotic expressions for $n - m \gg 1$, $m - k \gg 1$:

$$f_m^{(2)}(x_{0k}) - g_m^{(2)}(x_{0k}) = O\left(\frac{1}{n^{3/2}\sqrt{n-m}}\right).$$

Hence we have the following estimates:

$$\sum_{m=N_1+1}^{n}\left|f_m^{(2)}(x_{0k}) - g_m^{(2)}(x_{0k})\right| \leqslant \frac{C_2}{n^{\beta+0.5}}, \quad C_2 = \text{const} > 0,\ 0 < \beta < 0.5,$$

$$\left|\sum_{m=N_1+1}^{n}\Phi_2(x_m)\left[f_m^{(2)}(x_{0k}) - g_m^{(2)}(x_{0k})\right]\right| \leqslant \frac{M_2 C_2}{n^{\beta+0.5}}.$$

Collecting all the estimates together we obtain that for $k = 1, \ldots, n_1$

$$|F(x_{0k}) - G(x_{0k})| \leqslant \frac{2A_1B_1}{k^{\beta}n^{\alpha_1-0.5}} + \frac{M_2C_2}{n^{\beta+0.5}} + O\left(\frac{1}{k^{\beta}n^{\alpha_1+0.5}}\right). \tag{2.13}$$

In a similar way we obtain an estimate of the difference between the functions F and G at the points x_{0k} for $k = n_1 + 1, \ldots, n$

$$|F(x_{0k}) - G(x_{0k})| \leqslant \frac{2A_2B_2}{n^{\alpha_2+\beta-0.5}} + \frac{M_1C_1}{n^{\beta+0.5}} + O\left(\frac{1}{n^{\beta+\alpha_2+0.5}}\right), \tag{2.14}$$

where B_2 and C_1 are appropriate constants. In this case Eq. (2.12) becomes

$$\sqrt{n-N_2} - \frac{\sqrt{\tau_k}}{2}\ln\frac{\sqrt{n-N_2}+\sqrt{\tau_k}}{\sqrt{n-N_2}-\sqrt{\tau_k}} - \frac{1}{3}\sum_{m=N_2+1}^{n}\frac{(n-m+1)^{3/2}-(n-m)^{3/2}}{n-m-\tau_k+1-\mu_m} = 0,\ k = n_1+1, \ldots, n. \tag{2.15}$$

The solution of this equation enables one to determine the coefficients ν_k connected with τ_k by relation (2.9) for specified μ_m, or the coefficients μ_m for specified values of ν_k.

From estimates (2.13) and (2.14) we have:

THEOREM. Suppose the function $\gamma(x)$ for $x \in [0, 1]$ is represented in the form (1.2), where $\varphi(x)$ satisfies a Hölder condition with index α_1, $0.5 < \alpha_1 \leqslant 1$, for $x \in [0,\delta]$ and α_2 $0 < \alpha_2 \leqslant 1$, for $x \in (\delta, 1]$, $0 < \delta < 1$. A sequence of sets of points $x_{01}, \ldots, x_{0n}$ and $x_1, \ldots, x_n$ exists satisfying conditions (1.4), for which the function (1.6) converges uniformly to a Cauchy-type integral (1.1) at all points $x_{0k} \in [0, 1]$, when n increases without limit.

3. Analysis of Eqs. (2.12) and (2.15) shows that there is an infinite number of sets of points x_{0k}, x_m ($k, m = 1, \ldots, n$) satisfying Eqs. (2.12) and (2.15) and conditions (1.4). We will consider some of these.

TABLE 2

	n	ε_1	ε_2	ε_3	$\varepsilon_{n/2}$	ε_{n-2}	ε_{n-1}	ε_n
$\mu=1/4,\ \nu_k=3/4$	25 50 100	97 137 194	19 27 38	6,1 11 16	0,40 0,15 0,06	0,11 0,03 —0,01	0,09 0,02 —0,01	—4,3 —3,1 —2,2
Local approximation	25 50 100	2,2 1,8 1,7	0,66 0,39 0,17	0,34 0,12 —0,08	0,11 0,15 0,06	0,21 0,03 —0,01	0,18 0,02 —0,01	0,17 —0,08 —0,07

We will assume that all the coefficients $\mu_m = \mu = \text{const}$. Then, as $N \to \infty$ ($N = N_1$, $n - N_2$) the solutions of Eqs. (2.12) and (2.15) have the form

$$\nu_k(\mu) = \nu_k(0) + \mu,\ k = 1, \ldots, n. \tag{3.1}$$

Calculation shows that relation (3.1) is also satisfied in practice for a finite value of $N >> 1$. Hence, when $\mu_m = \text{const}$ it is sufficient to calculate the coefficients $\nu_k(0)$. The results of such a calculation for $N = 100$, rounded off to two decimal places, are shown in Table 1.

A calculation with $N = 1000$ gives the same results. According to Eqs. (1.5) and (3.1) the coordinates of the discrete vortices and the control points are given by the expressions

$$x_m = (m - 1 + \mu)/n,\ x_{0k} = (k - 1 + \nu_k(0) + \mu)/n, \quad m,\ k = 1, \ldots, n. \tag{3.2}$$

Note that when $\mu > 1 - \nu_k(0)$ the control point lies outside the limits of the element $[c_{k-1}, c_k]$, $k = 1, 2, \ldots$.

An example of the calculation of the error in approximating the Cauchy-type integral (1.1) by Eq. (1.6) when $\gamma(x) = \sqrt{(1 - x)/x}$ for a uniform distribution of the discrete vortices in the interval [0, 1] is given in Table 2. The quantity

$$\varepsilon_k = [1 - G(x_{0k})/F(x_{0k})]\cdot 100\%,\ k = 1, \ldots, n.$$

The calculation was carried out using two methods of choosing the coordinates of the discrete vortices and the control points $\mu = 1/4$, $\nu_k = 3/4$ and the local approximation.

The first method is widely used in the method of discrete vortices [2-5], while the second is based on Eqs. (3.1), (3.2), and the data in Table 1. Note that the calculation using the second method for $\mu = 0$, 0.25, and 0.5 gave identical results, which are shown in Table 2.

Calculation showed that in the middle part of the interval [0, 1] the error in approximating integral (1.1) by Eq. (1.6) is practically the same in both methods. In the region of the ends of the interval the first method gives a considerable error, which close to the end $x = 0$ increases as n increases. The local approximation with the values of n considered reduces the error in evaluating the integral (1.1) by two orders of magnitude in the region of the ends of the interval, and this error decreases as n increases.

Another example of the local approximation is the theoretical scheme proposed in [9]. According to this scheme, the discrete vortices are placed at the "center of gravity" of the vortex layer in each section $[c_{m-1}, c_m]$, while the control points are chosen, as additional analysis shows, depending on the solution of Eqs. (2.12) and (2.15).

The above examples illustrate the considerable possibilities of local approximation of a vortex layer by a system of discrete vortices. The main feature of this approximation is the solution of the problem for a certain part of the vortex layer irespective of the effect of its remaining part.

In this connection the results obtained can be used to approximate vortex layers with a different form of behavior of the intensity $\gamma(x)$ in the region of the ends of the layer. However, in this case it must be borne in mind that the number of control points and their position depend on the form of the function $\gamma(x)$. For example, in the case of the function $\gamma(x) = \sqrt{x(1 - x)}\varphi(x)$, bounded at both ends of the vortex layer, uniform convergence of the function (1.6) to the integral (1.1) occurs at $n + 1$ points for n discrete vortices situated in the middle of each element $[c_{m-1}, c_m]$ of the vortex layer. The coordinates of the control points are defined in the region of the ends of the interval [0, 1] by the solution of Eq. (2.15). According to the data given in Table 1

$$x_{01} = 0.12/n,\ x_{0k} = (k-1)/n,\ k = 2, \ldots, n,\ x_{0\,n+1} = 1 - x_{01}.$$

For the function $\gamma(x) = \varphi(x)/\sqrt{x(1-x)}$, not bounded at both ends of the interval [0, 1], uniform convergence of function (1.6) to integral (1.1) occurs at $n - 1$ points. In this case the discrete vortices should be again arranged in the middle of each element, while the coordinates of the control points

$$x_{0k} = (k - 0.5 + \nu_k(0))/n,\ k = 1, \ldots, n_1,\ x_{0k} = (k + 0.5 - \nu_{n-k}(0))/n,\ k = n_1 + 1, \ldots, n-1,$$

where the coefficients $\nu_k(0)$ are found from Table 1.

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THE WAKE BEHIND THE BEARING BODY IN A VISCOUS LIQUID

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1. Vortex Sheet. Suppose the bearing body (a wing of finite span) is loaded in a uniform flow of incompressible liquid. Assuming that the viscosity can be neglected, the most effective method of calculating the inductive resistance of the wing is as follows. The bearing surface and the contact interface formed behind it, and passing through which the tangential component of the velocity of vector is removed, are replaced by a system of attached and free vortices. The simplest version of the method operates in all with one connected vortex, which simulates the wing, and a pair of free vortices trailing from its ends. This system is sometimes called a horseshoe-shaped vortex, and it gives a lifting force (and circulation) that is constant over the whole area of the bearing surface, falling suddenly to zero at the ends of the wing. This scheme is, of course, only a rough approximation to the actual picture of the flow.

For a more accurate description of the flow field we must start from the fact that the lifting force changes over the length of the bearing surface, falling smoothly to zero at its end sections. The circulation also changes over the span of the wing, and from each point of its trailing edge a free vortex runs off which then moves downward with respect to the flow. This system of free vortices forms a vortex sheet. Although these representations were developed long ago [1, 2] they have not been of any value up to now [3]. In recent years to calculate the self-induced motion of vortex filaments the method of joining the external and

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